

Comparison Cones for Multiparameter Eigenvalue Problems

PAUL BINDING* AND PATRICK J. BROWNE†

*Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta,
Canada T2N 1N4*

Submitted by Ky Fan

We consider the multiparameter eigenvalue problem $(T_r + \sum_{s=1}^k \lambda_s V_{rs})x_r = 0$, $x_r \neq 0$, $1 \leq r \leq k$, where T_r and V_{rs} are self-adjoint linear operators on Hilbert spaces H_r , the V_{rs} being bounded. The problem may be posed in either $\bigoplus_{r=1}^k H_r$ or $\bigoplus_{r=1}^k H_r$ and we develop variational approaches for both settings. We explore the rôles played in both settings by $C = \{\lambda \in \mathbb{R}^k \mid \sum_{s=1}^k \lambda_s (V_{rs}x_r, x_r) \leq 0 \text{ for some non-zero } x_r \in H_r, 1 \leq r \leq k\}$ and related cones in \mathbb{R}^k . We also compare certain geometrical conditions on C with analytical definiteness conditions already in the literature.

1. INTRODUCTION

Recently [7, 8] we have investigated the multiparameter system of equations

$$\begin{aligned} W_r(\lambda)x_r &= 0, & x_r &\neq 0, \\ W_r(\lambda) &= T_r + \sum_{s=1}^k \lambda_s V_{rs}, & 1 \leq r \leq k, \end{aligned} \tag{1.1}$$

using variational techniques. Here $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$ while for each r , T_r , and V_{rs} , $1 \leq s \leq k$, are self-adjoint linear operators on a Hilbert space H_r . We restrict ourselves to the case in which the V_{rs} are bounded and the T_r bounded below with compact resolvent. We shall call the eigenvalue problem (1.1) a *k-parameter problem*. The monograph of Sleeman [13] is a compendium of recent developments in multiparameter spectral theory.

Problems of this type arise in various contexts connected with differential and difference equations. For example, separation of variables for a partial differential equation leads to systems where the H_r are L^2 spaces, the T_r are

* Research supported in part by NSERC Grant No. A9071

† Research supported in part by NSERC Grant No. A9073.

differential operators and the V_{rs} are multiplications by continuous functions. In the simplest cases $V_{rs} = 0$ whenever $r \neq s$, and we then obtain k separate 1-parameter problems for which the literature is vast. Sturm–Liouville theory falls into this category. Genuine k -parameter problems do arise in applied mathematics although they are somewhat less studied—see, however, [3].

Further examples can be found in polynomial eigenvalue problems of the form $(T + \sum_{s=1}^k \lambda^s V_s)x = 0$ and also in multi-point boundary value problems for ordinary differential equations. As Sleeman [13] and Arscott [1] show, such problems can often be profitably reformulated in a k -parameter form. Although the literature on these problems is over a century old, the abstract formulation (1.1) has been used only since the unifying work of Atkinson [2, 3, 4].

Our aim here is to summarize, and in certain ways, to complete our previous analysis [7, 8] of (1.1) from the variational point of view. Broadly speaking, we shall discuss results of a geometrical nature which do and do not carry over from $k=1$ to $k=2$ and from $k=2$ to $k \geq 3$. We shall consider to what extent the existence of solutions λ and x_r of (1.1) can be established via maximinima of generalized Rayleigh quotients in $H^{(1)} = \bigoplus_{r=1}^k H_r$ and in $H^{(2)} = \bigotimes_{r=1}^k H_r$. It turns out that $H^{(1)}$ is particularly useful for the existence and geometry of the spectrum Σ of eigenvalues λ , while $H^{(2)}$ is more appropriate for completeness relations of eigenvectors x_r and corresponding eigenvector expansion theorems.

The existence and comparison results will depend crucially on the cone $C \subseteq \mathbb{R}^k$ defined by

$$C = \left\{ \lambda \in \mathbb{R}^k \left| \sum_{s=1}^k (V_{rs} x_r, x_r) \lambda_s \leq 0 \text{ for } 1 \leq r \leq k \text{ and for some } \right. \right. \\ \left. \left. x_r \in H_r, x_r \neq 0, 1 \leq r \leq k \right\}.$$

When $k=1$, C is a half-line if V is definite. Assuming a standard definiteness condition on the V_{rs} , we show that C remains convex and contained in a half-plane for $k=2$. If $k \geq 3$, C may be neither convex nor contained in a half-space. We shall discuss some ramifications of these results.

In Section 2 we summarize the $H^{(1)}$ approach as developed in [8]. Section 3 contains an examination of the rôle played by C and related cones in locating some eigenvalues in terms of others. In Section 4 we consider how certain geometric properties of C relate to a standard definiteness condition, [4], in the cases $k=1, 2, 3$. Section 5 contains some implications for (1.1). In Section 6 we set up some of the $H^{(2)}$ framework and compare a geometrical property of C with another definiteness condition, [11]. Finally, in Section 7,

we develop the finite dimensional H^∞ approach [7] a little in the infinite dimensional case, again taking into account the analysis of Section 3.

2. THE DIRECT SUM APPROACH

Here we work with (1.1) directly and set $x = (x_1, \dots, x_k) \in H^\oplus$. We shall use λ to denote the point $(\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$. This practice will be used consistently for vectors and vector valued functions. We also introduce the notation U_r for the unit sphere of H_r and u_r for a typical element of U_r , so that $\|u_r\| = 1$. For $u = (u_1, \dots, u_k)$ with $u_r \in U_r \cap \mathcal{D}(T_r)$ we write $t(u) = (t_1(u), \dots, t_k(u))$, where $t_r(u) = (T_r u_r, u_r)$.

Our first boundedness assumption is that all the T_r are bounded below; that is we assume the existence of $\alpha \in \mathbb{R}^k$ so that $t(u) \geq \alpha$ for each u . Here, and throughout, the componentwise partial order is used in \mathbb{R}^k .

We can now claim that each $W_r(\lambda)$ has compact resolvent and is bounded below, and, as such, has a spectrum consisting entirely of eigenvalues:

$$\rho_r^0(\lambda) \leq \rho_r^1(\lambda) \leq \dots$$

each of finite multiplicity, accumulating only at ∞ , and obtainable via the minimax principle:

$$\rho_r^i(\lambda) = \text{Max}\{\text{Min}\{(W_r(\lambda) u_r, u_r) \mid u_r \in U_r \cap \mathcal{D}(T_r), (u_r, y_j) = 0\} \mid y_j \in H_r, 1 \leq j \leq i\}, \quad i \geq 0, \quad 1 \leq r \leq k. \quad (2.1)$$

Details may be found in [8, Lemma 1 and Corollary].

The second boundedness assumption concerns the matrix $V(u)$ whose (r, s) -entry is $(V_{rs} u_r, u_r)$. We assume that there is a real number $\beta > 0$ so that

$$|\det V(u)| \geq \beta \quad (2.2)$$

for each choice of $u_r \in U_r$, $1 \leq r \leq k$.

Subject to these assumptions we now have the following basic existence result.

THEOREM 2.1 [8, Theorem 2]. *Corresponding to each multi-index $i = (i_1, \dots, i_k) \geq 0$, where each i_r is an integer, there is an eigenvalue $\lambda^i \in \mathbb{R}^k$ and an eigenvector x^i with $0 \neq x_r^i \in H_r$ so that $\rho_r^{i_r}(\lambda^i) = 0$ and $W_r(\lambda^i) x_r^i = 0$, $1 \leq r \leq k$.*

It is also easy to show that all solutions of (1.1) occur in this way. We shall now consider some geometrical implications of the analysis thus far.

Let P_r^i, Z_r^i, N_r^i be the sets of $\lambda \in \mathbb{R}^k$, where $\rho_r^i(\lambda)$ is positive, zero, negative, respectively. Note that $j \geq i$ implies $P_r^i \subseteq P_r^j$ and $N_r^i \supseteq N_r^j$. Further, $\bigcup_{i=0}^\infty P_r^i = \mathbb{R}^k$ and $\bigcap_{i=0}^\infty N_r^i = \emptyset$ for each $r = 1, \dots, k$. Let

$$\dim H_r = 1 + d_r, \quad 1 \leq r \leq k.$$

We admit the possibility that $d_r = \infty$ and we set $P_r^\infty = \mathbb{R}^k, Z_r^\infty = N_r^\infty = \emptyset$. In general, the P_r^i, Z_r^i, N_r^i provide a partition of \mathbb{R}^k for each r —in fact P_r^i and N_r^i have common boundary Z_r^i . As we vary r , the P_r^i, N_r^i form a “patchwork” bordered by the Z_r^i with eigenvalues λ^i at the “corners” of these patches. More precisely, $\{\lambda^i\} = \bigcap_{r=1}^k Z_r^i$ —see [8, Sect. 4]. We shall illustrate these ideas with an example shortly.

We define the cone C by

$$C = \{\lambda \in \mathbb{R}^k \mid V(u)\lambda \leq 0 \text{ for some } u_r \in U_r, 1 \leq r \leq k\}. \quad (2.3)$$

Setting $N^i = \bigcap_{r=1}^k \bar{N}_r^i$ and Σ equal to the set of all eigenvalues λ^i we have

THEOREM 2.2 [8, Theorem 5]. *If $j \geq i$ then $N^j \subseteq \lambda^i + C$. In particular $\lambda^j \in \lambda^i + C$ and $\Sigma \subseteq \lambda^0 + C$.*

We remark that our proofs of these two theorems depend heavily on the minimax principle (2.1).

3. THE LOCATION OF EIGENVALUES

For the remainder of the paper, σ will denote any element of \mathbb{R}^k such that $\sigma_r = \pm 1, 1 \leq r \leq k$. Each such σ defines a partial order \leq_σ on \mathbb{R}^k by

$$a \leq b \quad \text{if, and only if,} \quad \sigma_r a_r \leq \sigma_r b_r, \quad 1 \leq r \leq k.$$

The remarks in Section 2 concerning the Cone C carry over without difficulty to the cones C_σ , where

$$C_\sigma = \{\lambda \in \mathbb{R}^k \mid V(u)\lambda \leq_\sigma 0 \text{ for some } u_r \in U_r, 1 \leq r \leq k\}. \quad (3.1)$$

In particular, the spectrum satisfies

$$\Sigma \subseteq \lambda^i + C_\sigma, \quad (3.2)$$

where (i, σ) is such that $(i_r, \sigma_r) = (0, 1)$ or $(d_r, -1)$ —the second alternative applies only if $d_r < \infty$ of course. Note that these “bounding” cones C_σ are independent not only of i but also of the operators $T_r, 1 \leq r \leq k$, and are determined completely by the array of operators $V_{rs}, 1 \leq r, s \leq k$.

We also note that any pair of multi-indices \mathbf{i}, \mathbf{j} is related by $\mathbf{i} \leq_{\sigma} \mathbf{j}$ for at least one σ , so

$$\lambda^{\mathbf{j}} \in \lambda^{\mathbf{i}} + C_{\sigma}. \quad (3.3)$$

In particular, if $0 < j_r < d_r$ for each r , then $\lambda^{\mathbf{j}}$ is surrounded by $2k$ neighbours $\lambda^{\mathbf{j}}$, where $j_r = i_r$ except for one r (say r_0) and $|j_{r_0} - i_{r_0}| = 1$. Thus (3.3) shows that $\lambda^{\mathbf{j}}$ is contained in $k2^k$ "neighbouring cones" which can thus be used to estimate $\lambda^{\mathbf{j}}$ using only the $\lambda^{\mathbf{i}}$ and the V_{rs} .

We now give a simple example for $k = 2$ to demonstrate the above ideas.

EXAMPLE 3.1. Let $H_1 = H_2 = \mathbb{C}^3$ and, in terms of matrices,

$$\begin{aligned} T_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, & V_{11} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & V_{12} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\ T_2 &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, & V_{21} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & V_{22} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

It is easily seen that our boundedness assumptions on \mathbf{t} and $\det V$ (2.2) are satisfied. Evidently $d_1 = d_2 = 2$ and the nine $\lambda^{\mathbf{i}}$, $\mathbf{0} \leq \mathbf{i} \leq (2, 2)$ are displayed with the $Z^{\mathbf{i}}$ in Manutract 1—observe that $\lambda^{\mathbf{0}}$ is a double eigenvalue. The four cones C_{σ} each have included angle $3\pi/4$.

Manutract 2 illustrates (3.2). In this case the intersection of the four "bounding" cones is the convex hull of Σ . Finally, manutract 3 illustrates (3.3) for $\mathbf{j} = (1, 1)$. Only four of the "neighbouring" cones are needed to locate $\lambda^{\mathbf{j}}$ exactly in this particular case.

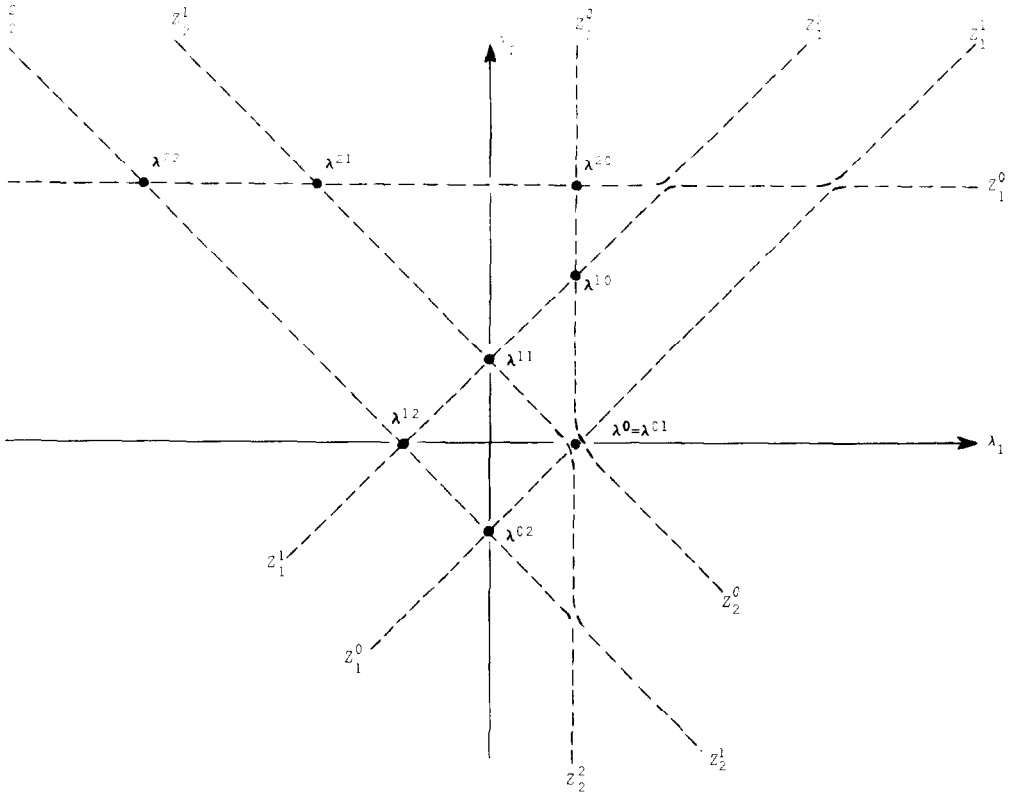
4. THE CONES C_{σ}

We shall now study the sets C_{σ} defined by (3.1). We observe initially that C_{σ} is a cone (that is $\varepsilon C_{\sigma} \subseteq C_{\sigma}$ for all $\varepsilon \geq 0$), and that $C_{-\sigma} = -C_{\sigma}$. For notational reasons we shall fix σ and suppress it when confusion cannot arise.

LEMMA 4.1. *If (2.2) holds then C_{σ} contains no line.*

Proof. If C_{σ} contains a line $\mathbb{R}\lambda$ with $\lambda \neq \mathbf{0}$, then by the above observation $\lambda \in C_{\sigma} \cap C_{-\sigma}$ and hence for some u, u' ,

$$V(u)\lambda \leq_{\sigma} \mathbf{0} \leq_{\sigma} V(u')\lambda.$$



MANUFRACT 1

Now $V(u)$ is continuous in u ; so applying the intermediate value theorem we obtain $V(u'')\lambda = 0$ for some u'' . Condition (2.2) now yields $\lambda = 0$ and this contradiction completes the proof.

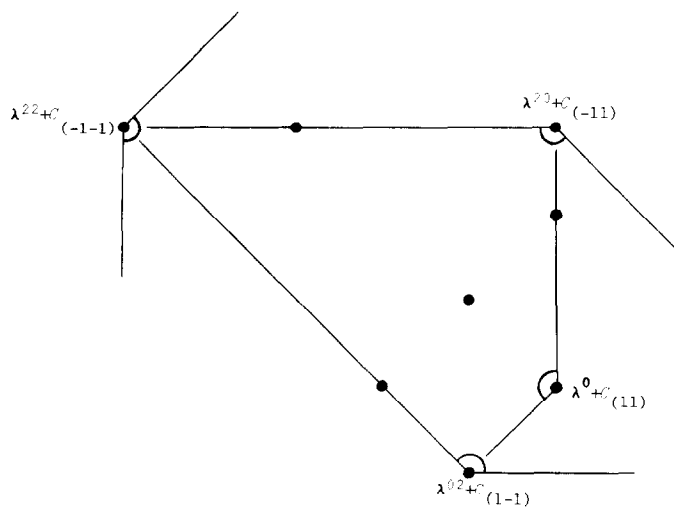
We may also put these observations in a form directly applicable to the comparison of eigenvalues—see (3.2), (3.3). We define a relation ρ on \mathbb{R}^k by

$$\lambda \rho \mu \quad \text{if, and only if,} \quad \lambda - \mu \in C.$$

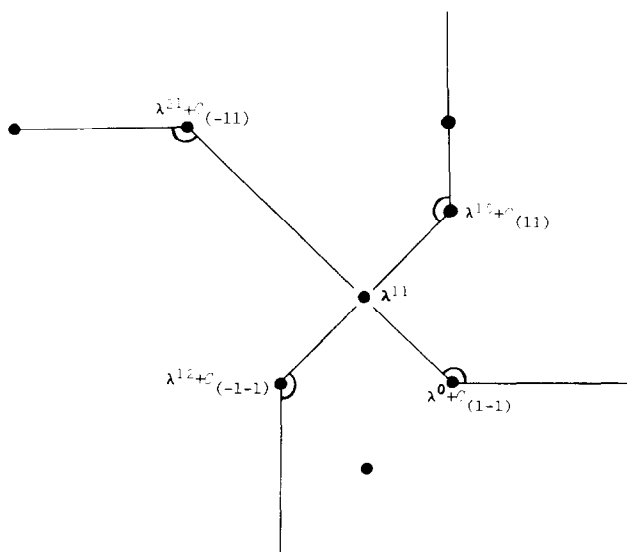
Then ρ is reflexive (that is, $\lambda \rho \lambda$) because C is a cone. Further, Lemma 4.1 shows that $\lambda \rho \mu$ and $\mu \rho \lambda$ imply $\lambda = \mu$. Accordingly, we have

COROLLARY 4.2. ρ is a partial order on \mathbb{R}^k if, and only if, C is convex.

We now require the following result [5, Lemma 4].



MANUTRACT 2



MANUTRACT 3

LEMMA 4.3. For some $\alpha > 0$ we have

$$\alpha \delta(u) \leq \inf_{\lambda \neq 0} \|V(u)\lambda\|/\|\lambda\| \leq \delta(u)^{1/k},$$

where $\delta(u) = \det V(u)$.

We shall also require further notation. Closure of a set in \mathbb{R}^k will be denoted by the customary bar. We define $V_r(u_r) \in \mathbb{R}^k$ to be the r th row of $V(u)$ —here $u_r \in U_r$. Finally, we set

$$\begin{aligned} C_r &= \{\mathbf{c} \in \mathbb{R}^k \mid \exists \mathbf{v} \in V_r(U_r), \mathbf{v}^T \mathbf{c} \leq 0\}, \\ C_r^+ &= \{\mathbf{c} \in \mathbb{R}^k \mid \exists \mathbf{v} \in \overline{V_r(U_r)}, \mathbf{v}^T \mathbf{c} \leq 0\}, \\ C &= \bigcap_{r=1}^k C_r, \quad C^+ = \bigcap_{r=1}^k C_r^+, \\ Q &= \text{Convex hull of } C^+. \end{aligned}$$

THEOREM 4.4. Condition (2.2) holds if, and only if, C_+ contains no line.

Proof. Suppose that (2.2) holds and that C^+ contains a line $\mathbb{R}\lambda$, $\lambda \neq 0$. As in Lemma 4.1, we obtain sequences $u_r'' \in U_r$ so that

$$\limsup_{n \rightarrow \infty} V_r(u_r'')^T \lambda \leq 0, \quad 1 \leq r \leq k, \quad (4.1)$$

and further sequences $u_r'^n \in U_r$ so that

$$\liminf_{n \rightarrow \infty} V_r(u_r'^n)^T \lambda \geq 0, \quad 1 \leq r \leq k. \quad (4.2)$$

If both inequalities (4.1), (4.2) are strict for some r we argue as in Lemma 4.1 and obtain $V_r(u_r'')^T \lambda = 0$. If not, then we obtain sequences $u_r^{*n} \in U_r$ so that

$$V_r(u_r^{*n})^T \lambda \rightarrow 0, \quad 1 \leq r \leq k.$$

Lemma 4.3 and the boundedness of the $V(u^{*n})$ now yield the contradiction $\lambda = 0$, so C^+ contains no line.

Conversely, suppose that (2.2) fails. Then from Lemma 4.3 we obtain sequences λ^n , $\|\lambda^n\| = 1$ and $u_r^n \in U_r$ so that

$$V(u^n)^T \lambda^n \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We may find a subsequence of λ^n convergent to say λ and deduce

$$V(u^n)^T \lambda \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

since the $V(u^n)$ are uniformly bounded. It follows that $\lambda \in C_r^+$ for each r , so C^+ contains the line $\mathbb{R}\lambda$. This completes the proof.

It will be seen below that C need be neither closed nor convex. It is not difficult to show that each C_r^+ is closed, so that C^+ is closed, but it need not be convex either.

DEFINITION 4.5. *The array $[V_{rs}]$ is proper if Q contains no line.*

COROLLARY 4.6. (i) *If $[V_{rs}]$ is proper then (2.2) holds.*

(ii) *C_r^+ is closed, whence $\overline{C_r} \subseteq C_r^+$.*

(iii) *If (2.2) holds then $C_r^+ = \overline{C_r}$.*

(iv) *If (2.2) holds and C is convex then $[V_{rs}]$ is proper.*

Proof. (i) This follows directly from Theorem 4.4.

(ii) Suppose $\mathbf{c}^n \in C_r^+$ and $\mathbf{c}^n \rightarrow \mathbf{c}$. Then there exist points $\mathbf{v}^n \in \overline{V_r(U_r)}$ with $\mathbf{v}^{nT}\mathbf{c}^n \leq 0$. Now $\|\mathbf{v}^n\| \leq \omega$ for some constant ω and so we may find a subsequence of \mathbf{v}^n converging to $\mathbf{v} \in \overline{V_r(U_r)}$. Then we have $\mathbf{v}^T\mathbf{c} \leq 0$ and so $\mathbf{c} \in C_r^+$.

(iii) We need to show $\overline{C_r} \supseteq C_r^+$. If (2.2) holds then $\mathbf{0} \notin \overline{V_r(U_r)}$, $1 \leq r \leq k$. Let $\mathbf{c} \in C_r^+$ with

$$\limsup_{n \rightarrow \infty} \mathbf{v}^{nT}\mathbf{c} \leq 0, \quad \mathbf{v}^n \in V_r(U_r).$$

Now define

$$\gamma^n = \mathbf{v}^{nT}\mathbf{c}^n$$

and

$$\begin{aligned} \mathbf{c}^n &= \mathbf{c} & \text{if } \gamma^n \leq 0, \\ \mathbf{c}_n &= \mathbf{c} - \gamma^n \mathbf{v}^n \|\mathbf{v}^n\|^{-2} & \text{if } \gamma^n > 0. \end{aligned}$$

It is evident that $\mathbf{c}^n \in C_r$ and further

$$\begin{aligned} \|\mathbf{c}^n - \mathbf{c}\| &= 0 & \text{if } \gamma^n \leq 0 \\ &= \gamma^n \|\mathbf{v}^n\|^{-1} & \text{if } \gamma^n > 0. \end{aligned}$$

Since $\limsup \gamma^n \leq 0$ and $\|\mathbf{v}^n\|$ is positively bounded below, we have $\mathbf{c}^n \rightarrow \mathbf{c}$, where $\mathbf{c} \in \overline{C_r}$, as required.

(iv) We shall show that $\overline{C} = C^+$, so that convexity of C will give $Q = C^+$ and the desired conclusion will follow from Theorem 4.4. In order to establish our claim, let $\mathbf{c} \in C^+$ and suppose $\mathbf{v}^{rT}\mathbf{c} \leq 0$, $r = 1, \dots, k$, where

$$\mathbf{v}^r = \lim_{n \rightarrow \infty} V_r(u_{nr}), \quad u_{nr} \in U_r.$$

If $\mathbf{v}^r \mathbf{c} = 0$ for each r , then $\det V = 0$, where the r th row of V is taken to be \mathbf{v}^r . This contradicts (2.2).

Now suppose $\mathbf{v}^r \mathbf{c} < 0$ for some r . Select a non-zero \mathbf{b} so that $\mathbf{v}^{sT} \mathbf{b} = 0$, $s \neq r$, $1 \leq s \leq k$ and set $\mathbf{c}'' = \mathbf{c} + \mathbf{b}/n$. Then for large n , $\mathbf{c}'' \in C_r^+ = \overline{C_r}$, $1 \leq r \leq k$ and $\mathbf{c}'' \rightarrow \mathbf{c}$. Thus $C^+ \subseteq \overline{C}$. The reverse inclusion follows from (ii).

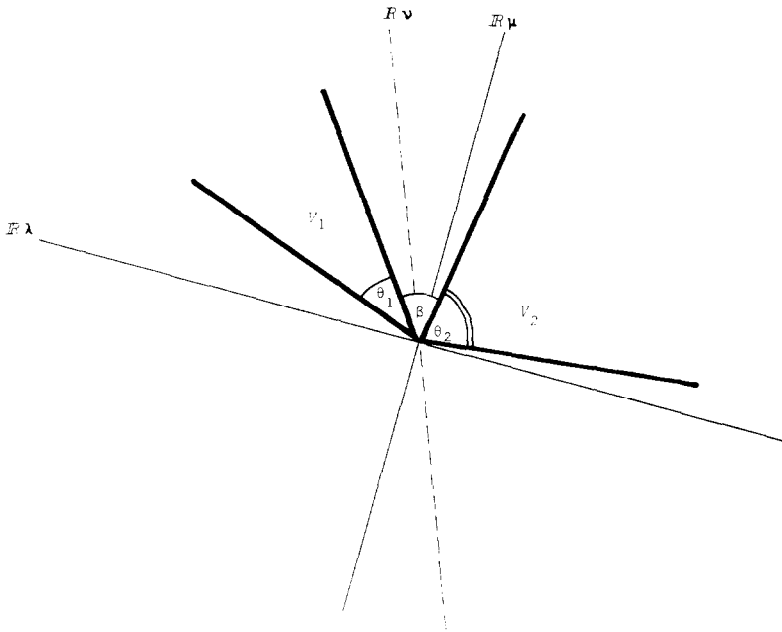
We note in passing that for $k=1$, C is closed and convex for either C equals \mathbb{R} or is a half-line containing 0 as an end point.

Next we define

$$V_r = \{\varepsilon \mathbf{v} \mid \mathbf{v} \in V_r(U_r), \varepsilon \geq 0\}.$$

THEOREM 4.7. *Let $k=2$ and assume (2.2) holds. Then C is convex and hence $[V_{rs}]$ is proper.*

Proof. From continuity and connectedness considerations, we see that V_1 and V_2 are cones corresponding to intervals of polar angle say of length θ_1, θ_2 , respectively. As in the proof of Theorem 4.4, the cones $\overline{V_1}, \overline{V_2}$ may be separated by a line $\mathbb{R}\boldsymbol{\mu}$, say, and likewise $\overline{V_1}, -\overline{V_2}$ may be separated by a line $\mathbb{R}\boldsymbol{\lambda}$. Finally, let the polar angle between V_1 and V_2 be β —see Manutract 4. The cones C_r now correspond to polar angle intervals of length $\pi + \theta_r$, $r=1, 2$ with overlap of $\pi - \beta$. Further since both $\overline{V_1}$ and $\overline{V_2}$ lie on the same



MANUTRACT 4

side of $\mathbb{R}\lambda$, $\beta + \theta_1 + \theta_2 < \pi$. From this we have $(\pi + \theta_1) + (\pi + \theta_2) - (\pi - \beta) < 2\pi$, so that the polar angle intervals for C_1, C_2 intersect in *one* interval of length $\pi - \beta$. This establishes the claim.

Before leaving the case $k = 2$ we point out that C may be convex without $|V_{rs}|$ being proper; i.e., without (2.2) holding. For example, if $H_1 = H_2$, $V_{11} = -V_{21} = I$ and $V_{12} = V_{22} = 0$ then C is the line $\lambda_1 = 0$.

The following is an example for which $|V_{rs}|$ is proper but C is not convex although (2.2) holds; naturally we require $k > 2$.

EXAMPLE 4.8. Let $H_1 = H_2 = H_3 = H$ —a separable Hilbert space with orthonormal basis e_1, e_2, \dots . Let I be the identity operator on H and let S be given by

$$Se_1 = e_1, \quad Se_n = 0, \quad n \geq 2.$$

Note that $u_r = \sum_{n=1}^{\infty} u_{rn} e_n \in U_r$ if, and only if, $\sum_{n=1}^{\infty} |u_{rn}|^2 = 1$, $r = 1, 2, 3$. Now let $V_{31} = -S$, $V_{32} = S - I$, $V_{rs} = \delta_{rs}I$ otherwise. Then (2.2) is easily seen to hold and for $r = 1, 2$, $C_r = \{\lambda | \lambda_r \leq 0\} = C_r^+$ since C_r is closed, and

$$\begin{aligned} C_3 &= \{\lambda | -\lambda_1 u_{31}^2 + \lambda_2 (u_{31}^2 - 1) + \lambda_3 \leq 0 \text{ for some } u_{31} \in [0, 1]\} \\ &= \{\lambda | \lambda_1 \geq \lambda_3 \text{ or } \lambda_2 \geq \lambda_3\} = C_3^+ \text{ since } C_3 \text{ is closed.} \end{aligned}$$

We see that $C = C^+$ and further $(-4, 0, -1)$ and $(0, -4, -1)$ both belong to C but $(-2, -2, -1) \notin C_3$. Thus C (or C^+) is not convex. On the other hand C is contained in the non-positive coördinate octant which is closed, convex and contains no line. The same then holds for Q and so $|V_{rs}|$ is proper.

5. FURTHER GEOMETRY OF THE SPECTRUM

We now consider how the geometry of C influences that of Σ . Recalling that if (2.2) holds then

$$\Sigma \subseteq \lambda^0 + C \tag{5.1}$$

we may conclude from our earlier results that for $k = 1$ or 2 , Σ is contained in the translate of a proper convex cone. In fact when $k = 1$, Σ is contained in a half-line with end point λ^0 , and when $k = 2$ we have

LEMMA 5.1. *Let $k = 2$ and assume (2.2) holds. Then there are distinct non-zero $\mu, v \in \mathbb{R}^2$ so that $\overline{V_1}$ and $\overline{V_2}$ are separated by both lines $\mathbb{R}\mu$ and $\mathbb{R}v$ while $\mu^T \lambda \leq 0$ and $v^T \lambda \leq 0$ for each $\lambda \in C$. In particular, $\mu^T \lambda^i \leq \mu^T \lambda^0$ and $v^T \lambda^i \leq v^T \lambda^0$ for each $i \geq 0$.*

This is geometrically obvious from Manutract 4. In general we have the following result, the bulk of which is contained in [4, Sect. 9.5, pp. 153–159].

LEMMA 5.2. *The following are equivalent.*

- (i) $[V_{rs}]$ is proper.
- (ii) There exist a non-zero $\omega \in \mathbb{R}^k$ and $\gamma > 0$ such that

$$\omega^T \lambda \leq -\gamma \|\lambda\| \quad \text{for all } \lambda \in C^+.$$

- (iii) There exist linearly independent $\omega^r \in \mathbb{R}^k$, $1 \leq r \leq k$, so that

$$\lambda^T \omega^r < 0 \quad \text{for all } \lambda \neq 0, \lambda \in C^+, \quad 1 \leq r \leq k.$$

We turn now to $k = 3$ and give an example in which (2.2) holds but for which $[V_{rs}]$ is not proper—in fact we shall have $Q = \mathbb{R}^3$. Using the setting of Example 4.8, we define the array $[V_{rs}]$ as

$$\begin{bmatrix} 5I & -I & -4S - I \\ 5I & 4S + I & 4S - 5I \\ 5I & 4S + I & I \end{bmatrix}.$$

Then for $u_r \in U_r$, $1 \leq r \leq 3$,

$$\det V(u) = 5[12 + 16(|u_{31}|^2 + |u_{11}u_{21}|^2 - |u_{21}u_{31}|^2 - |u_{31}u_{11}|^2)] \geq 60,$$

where we have used the fact that $0 \leq \varepsilon_r \leq 1$, $r = 1, 2, 3$ implies

$$\varepsilon_3 + \varepsilon_1 \varepsilon_2 - \varepsilon_2 \varepsilon_3 - \varepsilon_3 \varepsilon_1 \geq 0$$

—this is easily shown. Hence (2.2) holds.

We now claim that the vectors in \mathbb{R}^3

$$\xi^1 = (-1, 0, -5), \quad \xi^2 = (-1, 5, 0), \quad \xi^3 = (-1, -5, 5), \quad \xi^4 = (4, -5, 5) \quad (5.2)$$

all belong to C . This can be checked directly, but we shall give an indirect proof below via (5.1). Further

$$0 = \frac{1}{3}\xi^1 + \frac{1}{3}\xi^2 + \frac{2}{15}\xi^3 + \frac{1}{3}\xi^4$$

belongs to the interior of the tetrahedron A , formed by the ξ^j , $1 \leq j \leq 4$. Thus A contains an open ball $B \subseteq \mathbb{R}^3$ with centre 0 . Now we have $B \subset A \subset Q$ and since Q is a cone $\varepsilon B \subset Q$ for all $\varepsilon \geq 0$. It follows that $Q = \mathbb{R}^3$.

Next we supplement the above example with operators T_r chosen so that Σ reflects the geometry of C . Specifically, let $T_r = T$, $r = 1, 2, 3$, where

$$Te_1 = Te_2 = 0, \quad Te_n = ne_n, \quad n \geq 3.$$

Then T is self-adjoint with compact resolvent and is non-negative definite, so we may take our bound for $t(u)$ as $\alpha = 0$. Since $W_r(0) = T$, $r = 1, 2, 3$ we see

that $\lambda^i = 0$ for any i such that $i_r = 0$ or 1 , $r = 1, 2, 3$; the corresponding eigenvectors are (e_r, e_s, e_t) , $1 \leq r, s, t \leq 2$.

Elementary calculations also show that for $1 \leq s \leq 4$ and $j = 1, 2, \dots$, the vector $j\xi^s$ is an eigenvalue with eigenvector (e_l, e_m, e_n) . Here ξ^s is given by (5.2) and $(l, m, n) = (2, 1, 10j)$ if $s = 1$; $(10j, 2, 2)$ if $s = 2$; $(5j, 35j, 5j)$ if $s = 3$ and $(1, 10j, 1)$ if $s = 4$.

With $i = 0$ in (5.1) we obtain $\Sigma \subseteq C$. With $j = 1$ we have $\xi^s \in \Sigma \subseteq C$ justifying our earlier claim. It also follows by varying j that $\text{co } \Sigma = Q = \mathbb{R}^3$. Thus we have a problem (1.1) satisfying (2.2) with each T_r bounded below, whose spectrum is contained in no half-space and so has no supporting planes. We shall return to this in Section 7.

6. OPERATORS ON THE TENSOR PRODUCT

We shall now discuss the geometry of C within the tensor product setting. The latter is discussed in detail in [4, 10, 13]. Each operator V_{rs} induces an operator V_{rs}^+ on decomposables $u = u_1 \otimes \cdots \otimes u_k$ by

$$V_{rs}^+ u = u_1 \otimes \cdots \otimes u_{r-1} \otimes V_{rs} u_r \otimes u_{r+1} \otimes \cdots \otimes u_k$$

and this is extended to all of $H^{(\otimes)}$ by linearity and continuity. Induced operators from distinct rows commute, so the determinant

$$\Delta_0 = \det[V_{rs}^+]_{r,s=1}^k$$

is well defined as a bounded linear operator on $H^{(\otimes)}$. The cofactor of V_{rs}^+ in this expansion will be denoted by Δ_{0rs} , $1 \leq r, s \leq k$.

In $H^{(\otimes)}$ we use the induced inner product $(\cdot, \cdot) = \prod_{r=1}^k (\cdot, \cdot)_r$ and define quadratic forms

$$\delta_0(h) = (\Delta_0 h, h), \quad \delta_{0rs}(h) = (\Delta_{0rs} h, h), \quad 1 \leq r, s \leq k. \quad (6.1)$$

These forms are just real determinants when evaluated on decomposable tensors and, in particular,

$$\delta_0(u^{(\otimes)}) = \delta_0(u_1 \otimes \cdots \otimes u_k) = \det V(u).$$

This enables us to express (2.2) in the form

$$\delta_0(u^{(\otimes)}) \geq \beta \quad (6.2)$$

—recall that $\|u^{\otimes}\| = 1$. Note that (2.2) and continuity of the V_{rs} demand that $\det V(u)$ be of constant sign. Thus the difference between (2.2) and (6.2) reduces to a possible preliminary sign change.

Another definiteness condition has been used by Källström and Sleeman [11] and may be expressed as follows. For some non-zero $\omega \in \mathbb{R}^k$ and $\delta > 0$ the condition is that

$$\sum_{s=1}^k \omega_s \delta_{0rs}(u^{\otimes}) \geq \delta, \quad 1 \leq r \leq k. \quad (6.3)$$

THEOREM 6.1. $[V_{rs}]$ is proper if, and only if, (6.2) and (6.3) both hold.

Proof. In view of Corollary 4.6(i), it suffices to assume (6.2) and then establish that $[V_{rs}]$ is proper if, and only if, (6.3) holds.

From Lemma 5.2, $[V_{rs}]$ is proper if, and only if, for some $\omega \neq 0$ and $\gamma > 0$

$$\omega^T \lambda \leq -\gamma \|\lambda\|_1 \quad \text{for all } \lambda \in C^+. \quad (6.4)$$

Here $\|\cdot\|_1$ denotes the l_1 norm on \mathbb{R}^k . Now $\lambda \in C^+$ if, and only if, $V\lambda = \mathbf{q}$ for some $\mathbf{q} \leq 0$ and some matrix V with rows $\mathbf{v}^r \in \overline{V_r(U_r)}$, $1 \leq r \leq k$. Thus $[V_{rs}]$ is proper if, and only if,

$$\omega^T V^{-1} \mathbf{q} \leq -\zeta \|\mathbf{q}\|_1, \quad (6.5)$$

where $\zeta = \gamma / \sup \|V\|$ —the supremum to be taken over all such matrices V and $\|V\| = \sup_{r,s} |v_{rs}|$. Making use of (6.2) we may rewrite (6.5) as

$$\sum_{r,s=1}^k \omega_s \delta_{0rs}(u^{\otimes}) q_r \leq -\delta \|\mathbf{q}\|_1, \quad (6.6)$$

where $\delta = \beta\zeta$ and where we have specialized V to be of the form $V(u)$, $u_r \in U_r$, $1 \leq r \leq k$. Selecting \mathbf{q} to be each negative coördinate vector in turn, we then obtain (6.3).

For the reverse implication we have (6.3) implies (6.6) for all $\mathbf{q} \leq 0$ while (6.6) implies (6.5) with $\zeta = \delta / \sup \det V$ at least for matrices V of the form $V(u)$ as above. The general case for V is obtained as a limit of matrices $V(u)$. Now Lemma 4.3 shows that $\|\mathbf{q}\| \geq \eta \|\lambda\|$ for some $\eta > 0$ —independent of V —whenever $V\lambda = \mathbf{q}$. Thus (6.5) implies (6.4) with $\gamma = \eta\rho$, and the proof is complete.

When $k = 2$ we can conclude from Theorems 4.7 and 6.1 that (6.2) implies (6.3). This can be deduced directly—see [11] where it is shown that in general (6.3) is weaker than (6.2).

7. VARIATIONAL PRINCIPLES IN THE TENSOR PRODUCT

Recall that Δ_{0rs} is the cofactor of V_{rs}^+ in the expansion of $\Delta_0 = \det[V_{rs}^+]$. We now define as usual (cf. the references at the start of Section 6)

$$\Delta_s = - \sum_{r=1}^k T_r^+ \Delta_{0rs},$$

where T_r^+ is induced by T_r (by linearity: T_r^+ is unbounded if $d_r = \infty$). We induce $W_r^+(\lambda)$ similarly and note that all these operators are defined at least on $\mathcal{D} = \bigotimes_{r=1}^k \mathcal{D}(T_r)$, the algebraic tensor product of $\mathcal{D}(T_r)$, $1 \leq r \leq k$. Associated with Δ_s we have the quadratic form

$$\delta_s(h) = (\Delta_s h, h), \quad h \in \mathcal{D}, \quad 1 \leq s \leq k,$$

and we write

$$\delta(h) = (\delta_1(h), \dots, \delta_k(h)) \in \mathbb{R}^k.$$

Again the δ_s are just real $k \times k$ determinants when h is a decomposable tensor. Finally we define the vectorial range of the system (1.1) to be $\delta(U)$, where

$$U = \{h \in \mathcal{D} \mid \delta_0(h) = 1\}.$$

In the finite dimensional case, a result of Atkinson [4, Theorem 7.8.2] ensures that (6.2) extends to all of H^∞ , i.e., that

$$\delta_0(h) \geq \beta' \quad \text{for all } h \in H^\infty, \quad (h, h) = 1, \quad (7.1)$$

where $\beta' = \beta$, and so Δ_0^{-1} exists as a bounded operator on H^∞ . With respect to the new inner product $[\cdot, \cdot] = (\Delta_0 \cdot, \cdot)$ on H^∞ , the operators $\Gamma_r = \Delta_0^{-1} \Delta_r$ are pairwise commutative and self-adjoint, and the original problem (1.1) may be reformulated as the simultaneous eigenvalue problem

$$\Gamma_r x = \lambda_r x, \quad x \in H^\infty, \quad x \neq 0, \quad 1 \leq r \leq k.$$

We see that $\delta(U)$ is the range of "generalized Rayleigh quotients" $\delta(h)/\delta_0(h)$ and it turns out that $\delta(U) = \text{co } \Sigma$ [7, Lemma 2]. Further for any $\mathbf{a} \in \mathbb{R}^k$, $\mathbf{a}^T \delta(h)$ achieves a maximum value for $h \in U$. It follows that every "extreme" eigenvalue may be obtained via successive maximizations of functions $\mathbf{a}^T \delta$ over U for at most k distinct vectors \mathbf{a} . Full details can be found in [7, Theorem 1].

In the infinite dimensional case Binding [6] has investigated Atkinson's result mentioned above and has shown again that (6.2) implies (7.1) with $\beta' > 0$. However, further difficulties arise in extending the finite dimensional theory. We note that $H^\ominus = H^\infty$ when $k = 1$, so we shall begin with $k = 2$. In the following result, ≥ 0 denotes positive semi-definiteness of an operator.

LEMMA 7.1. *Let $S \geq 0$ be an Hermitian operator on a Hilbert space H_1 and let $T \geq 0$ be a self-adjoint operator on a Hilbert space H_2 . Then $S \otimes T \geq 0$ on $H_1 \otimes H_2$.*

Proof. If we define $S^\dagger = S \otimes I$ on $H_1 \otimes H_2$ and $T^\dagger = I \otimes T$ on $H_1 \otimes_a \mathscr{D}(T)$, then T^\dagger has a self-adjoint extension [12], $T^\dagger \geq 0$ and S^\dagger commutes with T^\dagger . Further S^\dagger has a self-adjoint square root $S' \geq 0$. Then for $x \in \mathscr{D}(T^\dagger)$,

$$(S \otimes T x, x) = (T^\dagger S^\dagger x, x) = (T^\dagger S' x, S' x) \geq 0$$

and the proof is complete.

We now recall the definition of λ^i and x^i from Theorem 2.1. We continue to use x^i to denote the decomposable eigenvector $\otimes_{r=1}^k x_r^i$ associated with λ^i .

THEOREM 7.2. *Let $k=2$ and assume (6.2). Define μ and ν as in Lemma 5.1. Then the upper bound of $\mu^T \delta$ on U is $\mu^T \lambda^0$ and is attained at x^0 . The same holds with μ replaced by ν .*

Proof. From Theorem 4.7 we see that $[V_{rs}]$ is proper. It is evident that we may replace ω by μ in the proof of Theorem 6.1 and so we may assume (6.3) in the form

$$\sum_{s=1}^2 \mu_s \delta_{0rs}(u^\otimes) \geq \delta, \quad r = 1, 2, \quad u^\otimes \in U. \quad (7.2)$$

Consider $r = 1$. Then (7.2) becomes

$$((\mu_1 V_{22}^\dagger - \mu_2 V_{21}^\dagger) u^\otimes, u^\otimes) \geq \delta, \quad (7.3)$$

and, bearing in mind Lemma 7.1 and the definition of V_{rs}^\dagger we may conclude

$$\sum_{s=1}^k \mu_s \Delta_{0rs} \geq 0 \quad (7.4)$$

for $r = 1$. A similar result holds for $r = 2$.

Now we have

$$\sum_{s=1}^k \mu_s \Delta_s = - \sum_{r,s=1}^k \mu_s W_r^\dagger(\lambda^0) \Delta_{0rs} + \sum_{q,r,s=1}^k \mu_s \Delta_{0rs} V_{rq}^\dagger \lambda_q^0 \quad (7.5)$$

Since $W_r(\mu^0) \geq 0$, $r = 1, 2$, Lemma 7.1, (7.4) and (7.5) give

$$\mu^T \delta(h) \leq \sum_{q,r,s=1}^k \mu_s \delta_{sq} \lambda_q^0 \delta_0(h).$$

Accordingly if $\delta_0(h) = 1$, we have

$$\mu^T \delta(h) \leq \mu^T \lambda^0. \quad (7.6)$$

The analysis is identical for μ replaced by ν and it is obvious that equality is achieved in (7.6) for $h = x^0$. This completes the proof.

Since μ and ν are linearly independent, it follows that this method can be used to determine λ^0 *ab initio*. We now turn to the case $k > 2$.

COROLLARY 7.3. *Assume (6.2) and (7.4) for $1 \leq r \leq k$. Then the upper bound of $\mu^T \delta$ on U is $\mu^T \lambda^0$ and is achieved at $\otimes_{r=1}^k x_r^0$.*

Proof. The proof is essentially as above using (7.4)–(7.6). The requisite properties of $W_r(\lambda^0)$ come from Theorem 2.1.

COROLLARY 7.4. *Assume that $[V_{rs}]$ is proper. Then the conclusion of Corollary 7.3 holds for k linearly independent vectors μ .*

Proof. Theorem 6.1 ensures that (6.2) and (6.3) both hold and the results of Binding [6] show that (6.2) and (6.3) extend to all vectors in H^∞ . This yields (7.4) in the strengthened version

$$\sum_{s=1}^k \mu_s A_{0rs} \geq \delta I, \quad \delta > 0$$

and now simple continuity arguments show that this holds for k linearly independent vectors μ . The proof now continues as before.

For the case $k = 3$, our example in Section 5 shows that it is possible to have $\text{co } \Sigma = \mathbb{R}^3$. On the other hand, we have shown that if F is the set of finite linear combinations of eigenvectors in \mathcal{L} then $\delta(U \cap F) = \text{co } \Sigma$, [8, Theorem 8]. Thus we have an example in which each T_r is bounded below but each Γ_r has a numerical range equal to \mathbb{R} . In particular, no non-trivial linear combination $a^T \delta$ is bounded below on U and hence there is no hope of characterizing eigenvalues via minima of $a^T \delta$.

REFERENCES

1. F. M. ARSCOTT, Two parameter eigenvalue problems in differential equations, *Proc. London Math. Soc.* **14** (1964), 459–470.
2. F. V. ATKINSON, “*Discrete and Continuous Boundary Value Problems*,” Academic Press, New York, 1964.
3. F. V. ATKINSON, Multiparameter spectral theory, *Bull. Amer. Math. Soc.* **74** (1968), 1–27.
4. F. V. ATKINSON, “*Multiparameter Eigenvalue Problems, Matrices and Compact Operators*,” Academic Press, New York, 1972.

5. P. A. BINDING, On the use of degree theory for nonlinear multiparameter eigenvalue problems, *J. Math. Anal. Appl.* **73** (1980), 381–391.
6. P. A. BINDING, Another positivity result for determinantal operators, *Proc. Roy. Soc. Edinburgh Sect. A*, in press.
7. P. A. BINDING AND P. J. BROWNE, A variational approach to multiparameter eigenvalue problems for matrices, *SIAM J. Math. Anal.* **8** (1977), 763–777.
8. P. A. BINDING AND P. J. BROWNE, A variational approach to multiparameter eigenvalue problems in Hilbert Space, *SIAM J. Math. Anal.* **9** (1978), 1054–1067.
9. P. A. BINDING AND P. J. BROWNE, Positivity results for determinantal operators, *Proc. Roy. Soc. Edinburgh Sect. A* **81** (1978), 267–271.
10. P. J. BROWNE, Abstract multiparameter theory, I, *J. Math. Anal. Appl.* **60** (1977), 259–273.
11. A. KÄLLSTRÖM AND B. D. SLEEMAN, A left-definite multiparameter eigenvalue problem in ordinary differential equations, *Proc. Roy. Soc. Edinburgh Sect. A* **74** (1974/5), 145–155.
12. E. PRUGOVEČKI, “*Quantum Mechanics in Hilbert Space*,” Academic Press, New York, 1971.
13. B. D. SLEEMAN, “*Multiparameter Spectral Theory in Hilbert Space*,” Pitman Press, London, 1978.